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Non-commutative/non-linear BPS equations without zero slope limit

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Abstract: It is widely believed that via the Seiberg-Witten map, the linearly realized BPS equation in the non-commutative space is related to the non-linearly realized BPS equation in the commutative space in the zero slope limit. We show that the relation also holds without taking the zero slope limit as is expected from the arguments of the BPS equation for the non-abelian Born-Infeld theory. This is regarded as an evidence for the relation between the two BPS equations. As a byproduct of our analysis, the non-linear instanton equation is solved exactly.

Keywords: Solitons Monopoles and Instantons, D-branes, Non-Commutative Geometry.

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Contents

1. Introduction and summary

Recently the string theory has been found to be more fertile than what it was thought previously. It contains many important concepts in physics including, among other things, non-commutativity. D-brane in the string theory with the background NS-NS 2-form B_{ij} has two effective theories: the ordinary Born-Infeld theory when the Pauli-Villars regularization is adopted [1] and the non-commutative Born-Infeld theory when the point-splitting regularization is adopted [2, 3]. Since the method of regularizations should not change the physical S-matrices, it was discussed in [4] that these two descriptions should be related by field redefinitions (called the Seiberg-Witten map).

This relation has also been explored from the classical solutions. In fact, many solitons and instantons were constructed in the non-commutative space [5]–[11] for investigating many interesting properties by themselves [12]–[16] and their relations to the commutative space [4], [17]–[20]. In the most cases except a few attempts [17, 20] the relation was discussed in the zero slope limit. In the non-commutative side the linearly realized BPS equation $\hat{F}^+ = 0$ is directly obtained from the BPS bound of the Yang-Mills theory [6] which is the zero slope limit of the Born-Infeld theory. Therefore it is widely believed that the linearly realized BPS equation in the non-commutative space (which we abbreviate to the non-commutative BPS equation) $\hat{F}^+ = 0$ is related to the non-linearly realized BPS equation in the commutative space (abbreviated to the non-linear BPS equation) in the zero slope limit $(F+B)^+/\operatorname{Pf}(F+B) = B^+/\operatorname{Pf} B$ by the Seiberg-Witten map.[1] This was implicitly pointed out in [4] by rewriting the non-linear BPS equation in the zero slope limit in terms of the open

[1] The non-linear BPS equation in the commutative space may be subject to derivative corrections of the string effective action. We shall discuss this issue in the final section.

string moduli: the open string metric G_{ij} and the non-commutativity parameter θ^{ij}, which originally appears in the non-commutative BPS equation and the Seiberg-Witten map. Though persuasive, it is still difficult to show the relation explicitly because the method used in [4] to connect two Born-Infeld actions cannot be applied to the case of the BPS equations.

On the other hand, it was shown in [21, 22] that although the linear BPS equation is obtained directly from the Yang-Mills theory, it also reproduces the equation of motion of the Born-Infeld theory if we adopt the symmetrized trace prescription [23]. Though this was shown in the non-abelian case, we can generalize the argument to the non-commutative case straightforwardly. The fact that the non-commutative BPS equation is unchanged in the zero slope limit implies that its commutative counterpart also remains the same under the zero slope limit because the Seiberg-Witten map $\delta\hat{A}_i = -1/4\delta\theta^{kl}(\hat{A}_k * (\partial_l\hat{A}_i + \hat{F}_{li}) + (\partial_l\hat{A}_i + \hat{F}_{li}) * \hat{A}_k)$ does not depend on the slope α'.

In this paper we rewrite the non-linear BPS equation in terms of the open string moduli without taking the zero slope limit. We show that, even though we do not take the zero slope limit, the non-linear BPS equation in the open string moduli takes completely the same form as that obtained in the zero slope limit. This calculation is also motivated by our previous work [20] where the non-linear BPS monopole solution without taking the zero slope limit was rewritten in terms of the open string moduli.[2]

This result shows that taking the zero slope limit is unnecessary in discussing the relation between the non-commutative BPS equation and non-linear BPS equation. More importantly, this result gives us a strong evidence for the conjecture that the non-commutative BPS equation is related to the non-linear BPS equation, because the invariant nature of the non-commutative BPS equation under the zero slope limit is perfectly reproduced in the commutative side.[3]

In the next section, we shall explicitly rewrite the non-linear BPS equation in terms of the open string moduli without taking the zero slope limit. And we shall discuss the physical implications and their applications in the final section.

2. Non-linear BPS equation in the open string moduli

In this section we shall rewrite the non-linear BPS equation in terms of the open string moduli. First let us recall the linear supersymmetries and non-linear super-

[2] In the monopole case [20] we only turned on the spatial B-field background. This corresponds to the special case of $\mathrm{Pf}\,B = 0$ in the instanton terminology and apparently leads to a singular BPS equation if we take the zero slope limit. That is why we tried to discuss without taking the zero slope limit in [20].

[3] Another interesting evidence is that both in the non-commutative side [7, 11] and the commutative side [20] the monopole moduli space is unchanged under the deformation of the non-commutativity parameter.

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symmetries of gauginos [24, 25, 23]:

$$\delta_{\rm L}\lambda_+ = \frac{1}{2\pi\alpha'}M^+_{ij}\sigma^{ij}\eta\,, \tag{2.1}$$

$$\delta_{\rm L}\lambda_- = \frac{1}{2\pi\alpha'}M^-_{ij}\bar\sigma^{ij}\bar\eta\,, \tag{2.2}$$

$$\delta_{\rm NL}\lambda_+ = \frac{1}{4\pi\alpha'}\left(1-{\rm Pf}\,M+\sqrt{1-{\rm Tr}\,M^2/2+({\rm Pf}\,M)^2}\right)\eta^*\,, \tag{2.3}$$

$$\delta_{\rm NL}\lambda_- = \frac{1}{4\pi\alpha'}\left(1+{\rm Pf}\,M+\sqrt{1-{\rm Tr}\,M^2/2+({\rm Pf}\,M)^2}\right)\bar\eta^*\,, \tag{2.4}$$

where M denotes

$$M = 2\pi\alpha'(F+B)\,, \tag{2.5}$$

with the field strength F_{ij} and the background NS-NS 2-form B_{ij}. Hereafter we shall set $2\pi\alpha' = 1$ for simplicity, however we can restore it on the dimensional ground. At the infinity the field strength vanishes and the combination of the unbroken supersymmetries is given as

$$B^+_{ij}\sigma^{ij}\eta + \frac{1}{2}\left(1-{\rm Pf}\,B+\sqrt{1-{\rm Tr}\,B^2/2+({\rm Pf}\,B)^2}\right)\eta^* = 0\,. \tag{2.6}$$

The non-linear BPS equation is the condition of preserving these supersymmetries:

$$\frac{M^+}{1-{\rm Pf}\,M+\sqrt{1-{\rm Tr}\,M^2/2+({\rm Pf}\,M)^2}} = \frac{B^+}{1-{\rm Pf}\,B+\sqrt{1-{\rm Tr}\,B^2/2+({\rm Pf}\,B)^2}}\,. \tag{2.7}$$

For rewriting this non-linear BPS equation (2.7) in terms of the open string moduli, we shall first rewrite it into a simpler form. First note from eq. (2.7), the matrix M^+ must be proportional to B^+:

$$M^+ = fB^+\,. \tag{2.8}$$

Rewriting eq. (2.7) into a scalar equation as

$$\begin{aligned} f\left(1-{\rm Pf}\,B+\sqrt{1-{\rm Tr}\,B^2/2+({\rm Pf}\,B)^2}\right) - (1-{\rm Pf}\,M) &= \\ = \sqrt{1-{\rm Tr}\,M^2/2+({\rm Pf}\,M)^2}\,,& \end{aligned} \tag{2.9}$$

by using eq. (2.8) and taking the square of eq. (2.9), eq. (2.7) is reduced to a much simpler form [26, 17]:

$$\frac{M^+}{1-{\rm Pf}\,M} = \frac{B^+}{1-{\rm Pf}\,B}\,. \tag{2.10}$$

Here we have used the following identities,

$${\rm Tr}\,M^2 = {\rm Tr}(M^+)^2 + {\rm Tr}(M^-)^2\,, \tag{2.11}$$

$$4\,{\rm Pf}\,M = -\,{\rm Tr}(M^+)^2 + {\rm Tr}(M^-)^2\,. \tag{2.12}$$

Note that if we further use the identity,

$$\mathrm{Pf}(F+B) = \mathrm{Pf}\,F + \mathrm{Pf}\,B - \mathrm{Tr}\,F\tilde{B}/2\,, \tag{2.13}$$

eq. (2.10) now reads

$$F^+(1-\mathrm{Pf}\,B) = B^+(\mathrm{Tr}\,F\tilde{B}/2 - \mathrm{Pf}\,F)\,. \tag{2.14}$$

Now let us proceed to rewriting eq. (2.14) in terms of the open string moduli: the open string metric G_{ij} and the non-commutativity parameter θ^{ij}. The open string moduli is related to the closed string moduli as [4]

$$\frac{1}{G} + \theta = \frac{1}{g+B}\,. \tag{2.15}$$

Since we adopt the flat metric for the closed string metric $g_{ij} = \delta_{ij}$, the open string metric G_{ij} and the non-commutativity parameter θ^{ij} are expressed in terms of the B-field:

$$G_{ij} = \delta_{ij} - (B^2)_{ij}\,, \tag{2.16}$$

$$\theta^{ij} = \frac{-B_{ij} - \tilde{B}_{ij}\,\mathrm{Pf}\,B}{\det(1+B)}\,. \tag{2.17}$$

Since in eq. (2.14) the self-dual projection appears, we also expect it to appear in the BPS equation in the open string moduli. As we know from [4] the easiest way to write down the self-dual projection is neither in the covariant frame nor in the contravariant one but in the local Lorentz frame. Hence we have to calculate

$$\underline{F}^+ = (E^t F E)^+\,, \tag{2.18}$$

$$\underline{\theta}^+ = \left(\frac{1}{E}\theta\frac{1}{E^t}\right)^+\,. \tag{2.19}$$

Here the vierbein is defined as $EGE^t = 1$. From the metric (2.16) we find that the vierbein is given as

$$E = (1+B)^{-1}\,. \tag{2.20}$$

In calculating the self-dual projection of the field strength, we shall go to a special frame where B has the canonical form as in [4]:

$$B = \begin{pmatrix} 0 & b_1 & 0 & 0 \\ -b_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 \\ 0 & 0 & -b_2 & 0 \end{pmatrix}\,. \tag{2.21}$$

In this frame $\underline{F}^+$ is given as

$$\underline{F}^+ = \left(\frac{1}{1-B}F\frac{1}{1+B}\right)^+ = \frac{1}{(1+b_1^2)(1+b_2^2)}\begin{pmatrix} 0 & f_1 & f_2 & f_3 \\ -f_1 & 0 & f_3 & -f_2 \\ -f_2 & -f_3 & 0 & f_1 \\ -f_3 & f_2 & -f_1 & 0 \end{pmatrix}, \tag{2.22}$$

where f_1, f_2 and f_3 denote

$$2f_1 = (1+b_2^2)F_{12} + (1+b_1^2)F_{34}\,, \tag{2.23}$$

$$2f_2 = (1-b_1b_2)(F_{13}-F_{24}) + (b_1+b_2)(F_{14}+F_{23})\,, \tag{2.24}$$

$$2f_3 = (1-b_1b_2)(F_{14}+F_{23}) - (b_1+b_2)(F_{13}+F_{24})\,. \tag{2.25}$$

We can easily identify terms proportional to F^+ and B^+, however there are still other terms to be identified as $B^+F^+ - F^+B^+$:

$$\Big(\frac{1}{1-B}F\frac{1}{1+B}\Big)^+ = \frac{(1-\mathrm{Pf}\,B)F^+ - (\mathrm{Tr}\,F\tilde{B})B^+/2 + (B^+F^+ - F^+B^+)}{\det(1+B)}\,. \tag{2.26}$$

On the other hand, $\underline{\theta}^+$ is much easier to calculate. We find

$$\underline{\theta}^+ = \Big((1+B)\theta(1-B)\Big)^+ = -B^+\,, \tag{2.27}$$

where we have used

$$B^3 = (\mathrm{Tr}\,B^2)B/2 + (\mathrm{Pf}\,B)\tilde{B}\,, \tag{2.28}$$

$$B\tilde{B}B = -(\mathrm{Pf}\,B)B\,, \tag{2.29}$$

which hold for any anti-symmetric 4×4 matrix B.

Since we are considering the non-linear BPS equation (2.7) which implies that F^+ is proportional to B^+ (2.8), it is possible to add $(B^+F^+ - F^+B^+)$ to eq. (2.14) freely because we have $B^+F^+ - F^+B^+ = 0$. Collecting all of our results and comparing them with eq. (2.14) we find it remains to rewrite $\mathrm{Pf}\,F$ into the local Lorentz frame:

$$\mathrm{Pf}\,\underline{F} = \frac{\mathrm{Pf}\,F}{\det(1+B)}\,. \tag{2.30}$$

Therefore our non-linear BPS equation (2.7) is finally rewritten as

$$\underline{F}^+ = \underline{\theta}^+\,\mathrm{Pf}\,\underline{F}\,. \tag{2.31}$$

Amazingly, this is the same form as that obtained in the zero slope limit [4].

3. Physical implications and further directions

First, our analysis in this paper is important in the conceptual sense. We have given another strong evidence for the fact that the non-commutative BPS equation is mapped to the non-linear BPS equation, because the invariant nature of the non-commutative BPS equation under the zero slope limit is perfectly reproduced in the commutative side. In fact, the non-linear BPS equation (2.31) without taking the zero slope limit has the same form as that obtained in the zero slope limit [4] when they are rewritten in terms of the open string moduli.

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Secondly, so far we have neglected the stringy derivative corrections to the Born-Infeld theory. Strictly speaking, it is the non-commutative and commutative Born-Infeld theory with derivative corrections that should be related by the Seiberg-Witten map, not the Born-Infeld theories themselves. Hence if we would like to discuss the relation between the two BPS equations, we have to see if the two BPS equations remain unchanged when the derivative corrections are taken into account. In the commutative side, the solutions of the linear BPS equation are also solutions of the Born-Infeld theory with corrections [27]. However we do not know a similar argument for the non-linear BPS equation. To clarify this point is an interesting direction. Our result (2.31) might give a clue to this question.

Thirdly, as a technical application this rewriting enables us to find an instanton solution to the non-linear BPS equation for a general constant B-field background without taking the zero slope limit. In [17] the solution is constructed under the condition $B^- = 0$, because otherwise the solution is very intricate. However, the non-linear BPS equation is now rewritten as (2.31), which is completely solved in [4]. Hence it is possible to read off the solution to the non-linear BPS equation directly:

$$A_i = \theta^+_{ij} x^j \cdot \frac{1}{4}\left(-1 + \sqrt{1 + \frac{32C}{R^4}} \right), \tag{3.1}$$

with $2\theta^+_{ij} = -(1 - \operatorname{Tr} B^2/2 + \operatorname{Pf} B)B_{ij} - (1 - \operatorname{Pf} B)\tilde{B}_{ij}$ and $R^2 = x^i(1 - B^2)_{ij}x^j$. Though the solution should be independent of θ^- as noted in [4], our solution (3.1) has a non-trivial dependence on B^-. In this way the commutative counterparts of the non-commutative abelian instanton and monopole [5, 11] with or without taking the zero slope limit are all constructed [4, 20].

Finally, since we have rewritten the non-linear BPS equation completely in terms of the open string moduli, we should expect the tension of the exact monopole solution found in [20] also has a trivial dependence on α'. However, it was discussed in [11] that this is not the case and there might be a discrepancy between the non-commutative and commutative viewpoints. It is very important to resolve this discrepancy.

Note added. After submitting this paper for the publication, the authors of [11] made a revision to resolve the discrepancy between the monopole tensions mentioned in section 3 of this paper and showed that they do not depend on α'.

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References

[1] A. Abouelsaood, C.G. Callan, C.R. Nappi and S.A. Yost, *Open strings in background gauge fields*, *Nucl. Phys.* **B 280** (1987) 599.

[2] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[3] M.R. Douglas and C. Hull, *D-branes and the noncommutative torus*, *J. High Energy Phys.* **02** (1998) 008 [hep-th/9711165].

[4] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[5] N. Nekrasov and A. Schwarz, *Instantons on noncommutative R^4 and $(2,0)$ superconformal six dimensional theory*, *Comm. Math. Phys.* **198** (1998) 689 [hep-th/9802068].

[6] K. Hashimoto, H. Hata and S. Moriyama, *Brane configuration from monopole solution in non-commutative super Yang-Mills theory*, *J. High Energy Phys.* **12** (1999) 021 [hep-th/9910196].

[7] D. Bak, *Deformed Nahm equation and a noncommutative BPS monopole*, *Phys. Lett.* **B 471** (1999) 149 [hep-th/9910135].

[8] H. Hata and S. Moriyama, *String junction from non-commutative super Yang-Mills theory*, *J. High Energy Phys.* **02** (2000) 011 [hep-th/0001135].

[9] R. Gopakumar, S. Minwalla and A. Strominger, *Noncommutative solitons*, *J. High Energy Phys.* **05** (2000) 020 [hep-th/0003160].

[10] S. Goto and H. Hata, *Noncommutative monopole at the second order in theta*, hep-th/0005101.

[11] D.J. Gross and N.A. Nekrasov, *Monopoles and strings in noncommutative gauge theory*, *J. High Energy Phys.* **07** (2000) 034 [hep-th/0005204].

[12] A. Hashimoto and K. Hashimoto, *Monopoles and dyons in non-commutative geometry*, *J. High Energy Phys.* **11** (1999) 005 [hep-th/9909202].

[13] H.W. Braden and N.A. Nekrasov, *Space-time foam from non-commutative instantons*, hep-th/9912019.

[14] K. Furuuchi, *Instantons on noncommutative R^4 and projection operators*, hep-th/9912047.

[15] K. Dasgupta, S. Mukhi and G. Rajesh, *Noncommutative tachyons*, *J. High Energy Phys.* **06** (2000) 022 [hep-th/0005006].

[16] J.A. Harvey, P. Kraus, F. Larsen and E.J. Martinec, *D-branes and strings as non-commutative solitons*, *J. High Energy Phys.* **07** (2000) 042 [hep-th/0005031].

[17] S. Terashima, U(1) *instanton in Born-Infeld action and noncommutative gauge theory*, *Phys. Lett.* **B 477** (2000) 292 [hep-th/9911245].

[18] D. Mateos, *Non-commutative vs. commutative descriptions of D-brane BIons*, *Nucl. Phys.* **B 577** (2000) 139 [hep-th/0002020].

[19] K. Hashimoto and T. Hirayama, *Branes and BPS configurations of noncommutative/commutative gauge theories*, hep-th/0002090.

[20] S. Moriyama, *Noncommutative monopole from nonlinear monopole*, *Phys. Lett.* **B 485** (2000) 278 [hep-th/0003231].

[21] A. Hashimoto, *The shape of branes pulled by strings*, *Phys. Rev.* **D 57** (1998) 6441 [hep-th/9711097].

[22] D. Brecher, *BPS states of the non-abelian Born-Infeld action*, *Phys. Lett.* **B 442** (1998) 117 [hep-th/9804180].

[23] A.A. Tseytlin, *Born-Infeld action, supersymmetry and string theory*, hep-th/9908105.

[24] J. Bagger and A. Galperin, *A new Goldstone multiplet for partially broken supersymmetry*, *Phys. Rev.* **D 55** (1997) 1091 [hep-th/9608177].

[25] S.V. Ketov, *A manifestly $N = 2$ supersymmetric Born-Infeld action*, *Mod. Phys. Lett.* **A 14** (1999) 501 [hep-th/9809121].

[26] M. Marino, R. Minasian, G. Moore and A. Strominger, *Nonlinear instantons from supersymmetric p-branes*, *J. High Energy Phys.* **01** (2000) 005 [hep-th/9911206].

[27] L. Thorlacius, *Born-Infeld string as a boundary conformal field theory*, *Phys. Rev. Lett.* **80** (1998) 1588 [hep-th/9710181].

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